Chapter 5. Linear Programming

Session 1. Formulation

Explain what is a linear relationship!!

Explain the difference between variables and parameters!!

Explain decision variables in Inventory Management

Example 1 *Pottery Problem*

There are 40 hours of labor and 120 pounds of clay available each day

	Resource Requirements		
Product	Labor (hr./unit)	Clay (lb./unit)	Revenue ($/unit)
Bowl	1	4	40
Mug	2	3	50

(Decision variables)

x_1 = number of bowls to produce

x_2 = number of mugs to produce

Maximize $Z = 40\,x_1 + 50\,x_2$

Subject to

$x_1 + 2x_2 \leq 40\ hr.$ (Labor constraint)

$4x_1 + 3x_2 \leq 120\ lb.$ (Clay constraint)

$x_1, x_2 \geq 0$

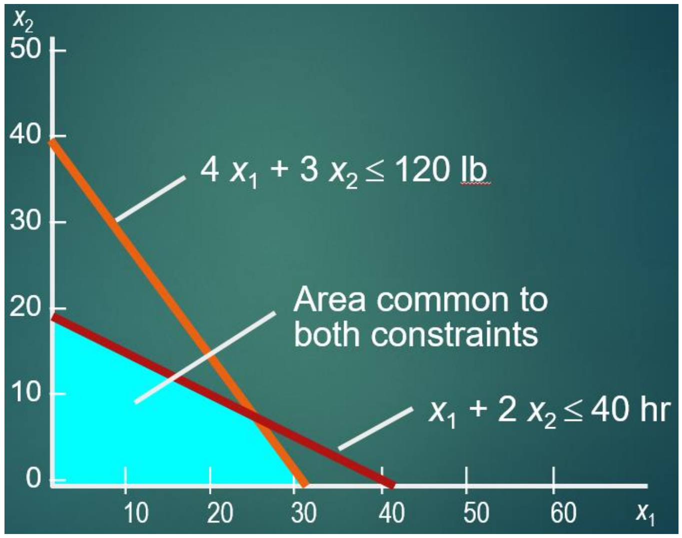
Solution is $x_1 = 24$ bowls, $x_2 = 8$ mugs, Revenue $= \$1{,}360$

<u>Graphical Solution Method</u>

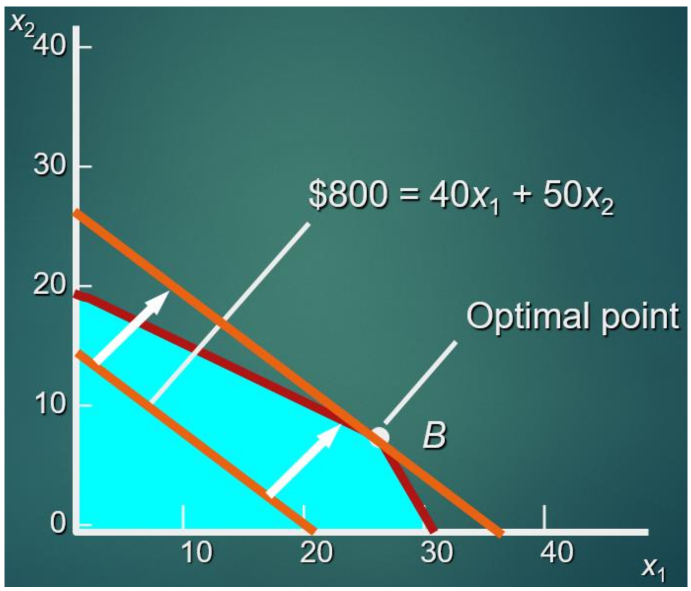
1. Plot model constraint on a set of coordinates in a plane

2. Identify the feasible solution space on the graph where all constraints are satisfied simultaneously

3. Plot objective function to find the point on boundary of this space that maximizes (or minimizes) value of objective function



Next, plot the objective function



Computing Optimal Values

Solve the equation system:

$$\begin{cases} x_1 + 2x_2 = 40 \\ 4x_1 + 3x_2 = 120 \end{cases}$$

$Z = \$50(24) + \$50(8) = \$1{,}360$

Next, let us solve another problem in a more standard manner

Glickman Electronics Example

Glickman Electronics company produces two products:

1) Glickman x-pod 2) Glickman BlueBerry

Both require

a) electronic work b) assembly

	Hours Required to Produce One Unit		
Department	X-PODS	BLUEBERRY	Available Hours This Week
Electronic	4	3	240
Assembly	2	1	100
Profit per unit	$7	$5	

We want to maximize the profit.

How to solve this problem?

Business Problem ⟹ (abstracted into) Math Problem⟹ Solve it with math tools (Operations Research) ⟹ Put the result back into the business background

The ability to "**Abstract**" a business problem into a math problem is one of business students' core competences. We have business problem – usually we have a trade off, something goes up, something will go down (inventory, if you want to stock, holding cost will increase…but the order cost/setup cost will decrease, then you will have quadratic curve) – we need to find a balance point to maximize the profits or minimize the cost. Then we abstract this business problem to a math problem, and solve it, this is what OM does. How to solve it? It is about math, usually we call it OR

1. Transfer a business context to a pure math problem – Formulation

The first step is the most important and usually most difficult.

2. Solve the math.

We usually have some regular/conventional method in this step. Nowadays' research focuses on this step because problems can be too difficult to solve. Researchers will think of some method to find a sub-optimal solution (the optimal solution will take hundreds of years)

Let's focus on the first step.

Summary of Formulation

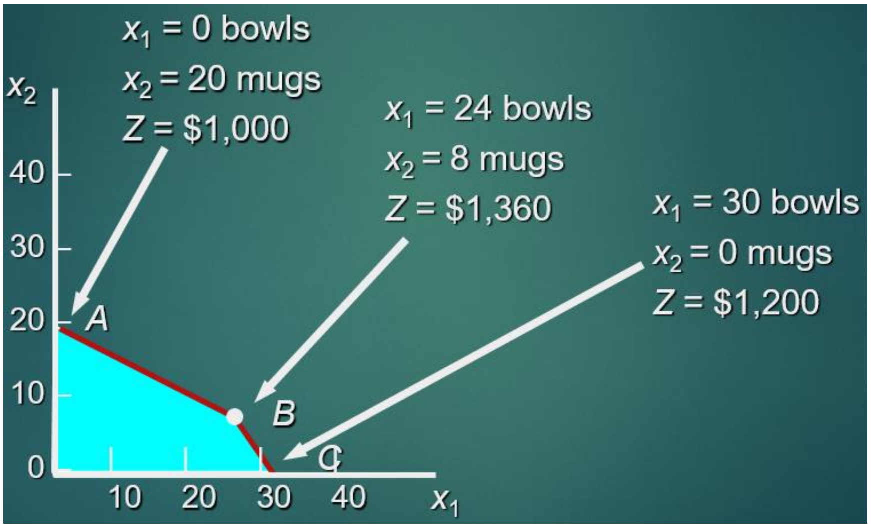
Model consisting of linear relationships representing a firm's objectives & resource constraints

Decision variables are mathematical symbols representing levels of activity of an operation

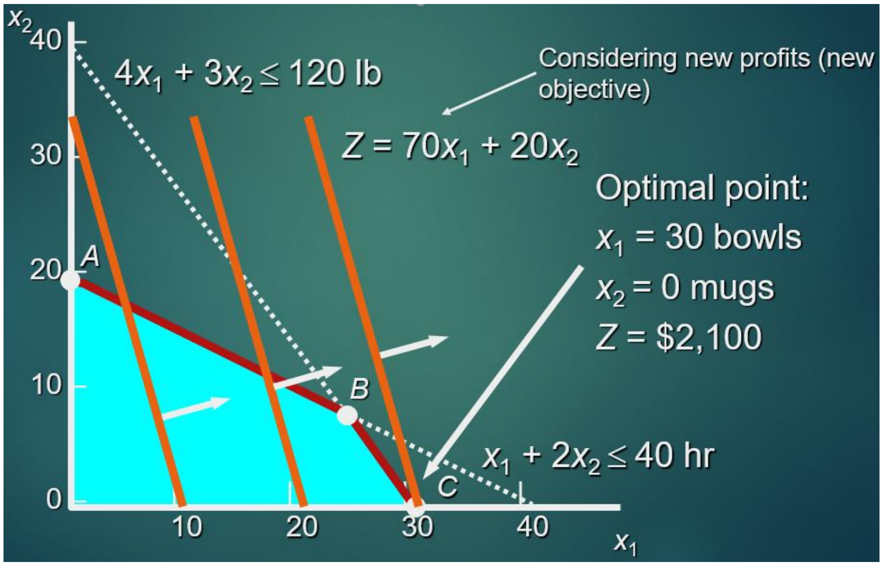
Objective function is a linear relationship reflecting objective of an operation

Constraint is a linear relationship representing a restriction on decision making

Extreme Corner Points



Objective Function Determines Optimal Solution



In mathematical optimization, the fundamental theorem of linear programming states, in a weak formulation, that the maxima and minima of a linear function over a convex polygonal region occur at the region's corners. Further, if an extreme value occurs at two corners, then it must also occur everywhere on the **line segment** between them. (From Wikipedia)

Suppose the slope of the objective function is the same with one of the constraints

Minimization Problem

Formulated and solved in much the same way as maximization problems

In the graphical approach an iso-cost line is used

The objective is to move the iso-cost line inwards until it reaches the lowest cost corner point

x_1 = number of tons of black-and-white chemical produced

x_2 = number of tons of color picture chemical produced

Minimize total cost $= 2{,}500x_1 + 3{,}000x_2$

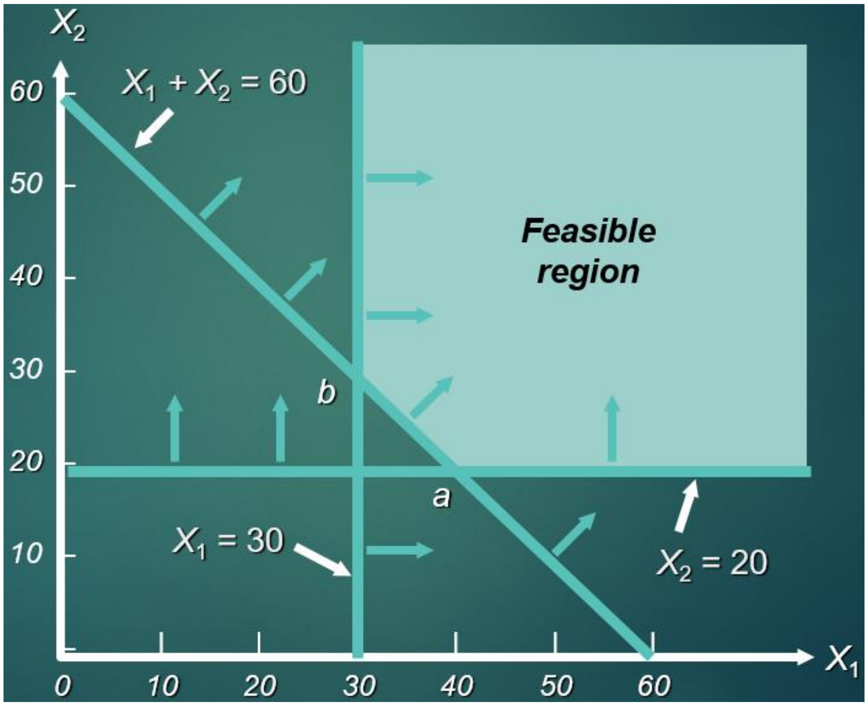
Subject to:

$x_1 \geq 30$ *tons of black-and-white chemical*

$x_2 \geq 20$ *tons of color chemical*

$x_1 + x_2 \geq 60$ *tons total*

$x_1, x_2 \geq \$0$ *nonnegativity requirements*



$Total\ cost\ at\ a = 2{,}500x_1 + 3{,}000x_2$

$= 2{,}500(40) + 3{,}000(20)$

$= \$160{,}000$

$Total\ cost\ at\ b = 2{,}500x_1 + 3{,}000x_2$

$= 2{,}500(30) + 3{,}000(30)$

$= \$165{,}000$

Lowest total cost is at point a

The Simplex Method

Real world problems are too complex to be solved using the graphical method

The simplex method is an algorithm for solving more complex problems

Developed by George Dantzig in the late 1940s

Most computer-based LP packages use the simplex method

Session 2. LP Applications – More examples

Diet Problem Example

Each cow needs at least:

64 oz. A,

80 oz. B,

16 oz. C,

128 oz. D

	Feed		
Ingredient	Stock X	Stock Y	Stock Z
A	3 oz.	2 oz.	4 oz.
B	2 oz.	3 oz.	1 oz.
C	1 oz.	0 oz.	2 oz.
D	6 oz.	8 oz.	4 oz.

<u>Prices</u>

Stock X: $0.02 / lb.

Stock Y: $0.04 / lb.

Stock Z: $0.025 / lb.

X_1 = number of pounds of stock X purchased per cow each month

X_2 = number of pounds of stock Y purchased per cow each month

X_3 = number of pounds of stock Z purchased per cow each month

Minimize cost $= .02x_1 + .04x_2 + .025x_3$

Ingredient A requirement: $3x_1 + 2x_2 + 4x_3 \geq 64$

Ingredient B requirement: $2x_1 + 3x_2 + 1x_3 \geq 80$

Ingredient C requirement: $1x_1 + 0x_2 + 2x_3 \geq 16$

Ingredient D requirement: $6x_1 + 8x_2 + 4x_3 \geq 128$

Stock Z limitation: $x_3 \leq 5$

$x_1, x_2, x_3 \geq 0$

Cheapest solution is to purchase 40 pounds of grain X at a cost of $0.80 per cow

Labor Scheduling Example

Time Period	Number of Tellers Required	Time Period	Number of Tellers Required
9 AM – 10 AM	10	1 PM – 2 PM	18
10 AM – 11 AM	12	2 PM – 3 PM	17
11 AM – Noon	14	3 PM – 4 PM	15
Noon – 1 PM	16	4 PM – 5 PM	10

<u>Other Requirements</u>:

At most 12 Full-time Tellers are available

Part time workers cannot exceed 50% of all hours required in a day

Part-time tellers make $24 for their four hour shift

Full-time tellers make $75 for their 8 hour shift

Half of full-time workers get lunch from 11-noon, the rest from noon-1pm

Define Decision Variables:

F = Full-time tellers

P_1 = Part-time tellers starting at 9 AM (leaving at 1 PM)

P_2 = Part-time tellers starting at 10 AM (leaving at 2 PM)

P_3 = Part-time tellers starting at 11 AM (leaving at 3 PM)

P_4 = Part-time tellers starting at noon (leaving at 4 PM)

P_5 = Part-time tellers starting at 1 PM (leaving at 5 PM)

Minimize total daily manpower cost $= \$75\mathrm{F} + \$24(\mathrm{P1} + \mathrm{P2} + \mathrm{P3} + \mathrm{P4} + \mathrm{P5})$

$F + P_1 \geq 10$ (9 AM - 10 AM needs)

$F + P_1 + P_2 \geq 12$ (10 AM - 11 AM needs)

$\frac{1}{2}F + P_1 + P_2 + P_3 \geq 14$ (11 AM - noon needs)

$\frac{1}{2}F + P_1 + P_2 + P_3 + P_4 \geq 16$ (Noon - 1 PM needs)

$F + P_2 + P_3 + P_4 + P_5 \geq 18$ (1 PM - 2 PM needs)

$F + P_3 + P_4 + P_5 \geq 17$ (2 PM - 3 PM needs)

$F + P_4 + P_5 \geq 15$ (3 PM - 7 PM needs)

$F + P_5 \geq 10$ (4 PM - 5 PM needs)

$F \leq 12$

$4(\mathrm{P}_1 + \mathrm{P}_2 + \mathrm{P}_3 + \mathrm{P}_4 + \mathrm{P}_5) \leq .50(10 + 12 + 14 + 16 + 18 + 17 + 15 + 10)$

$F, \mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3, \mathrm{P}_4, \mathrm{P}_5 \geq 0$

There are two alternate optimal solutions to this problem but both will cost $1,086 per day

Decision Variables	First Solution	Second Solution
F	10	10
P_1	0	6
P_2	7	1
P_3	2	2
P_4	2	2
P_5	3	3

Transportation Example

Cost:

From/To	Chicago	St. Louis	Cincinnati
Kansas City	6.00	8.00	10.00
Omaha	7.00	11.00	11.00
Des Moines	4.00	5.00	12.00

Supply/Demand

GRAIN ELEVATOR	Supply	Mill	Demand
Kansas City	150	Chicago	200
Omaha	175	St. Louis	100
Des Moines	275	Cincinnati	300

Formulation

a) Decision Variable

x_{ij} The volume transported from City i to City j

b) Constraints

$i = 1, \sum_{j=1}^{3} x_{1j} \leq 150$

$i = 2, \sum_{j=1}^{3} x_{2j} \leq 175$

$i = 3, \sum_{j=1}^{3} x_{3j} \leq 275$

$j = 1, \sum_{i=1}^{3} x_{i1} \geq 200$

$j = 2, \sum_{i=1}^{3} x_{i2} \geq 100$

$j = 3, \sum_{i=1}^{3} x_{i3} \geq 300$

$x_{ij} \geq 0 \; \forall i, j$

c) Objective Function

Min Z = $\sum_i \sum_j c_{ij} x_{ij}$, where c_{ij} means cost

Sensitivity Analysis

Very Common in Business Context

Intuition

We design a policy under a specific context to optimize business value. How we design the policy depends on the context. Sensitivity analysis is an approach to test how sensitive our policy depends on our context. Suppose there are two parameters in our context: temperature and pressure, if either of the two parameters changes a little, leading to our policy changing a lot, then our policy designed is not very robust/stable, which may not be very valuable. A good policy should be robust. We do not want to change our policy frequently, because frequent and large variation in the context means high risk.

Session 3. Integer Programming

Question: what if the solution can only be integers? e.g., number of people, number of products, whether to choose one case

Round to the nearest integers? No!

If we wish to ensure that decision variable values are integers rather than fractions, it is generally *not* good practice to simply round the solutions to the nearest integer values. The rounded solutions may not be optimal and, in fact, may not even be feasible.

Binary Variables

Special decision variables called **binary variables** that can only take on the values of 0 or 1. Binary variables allow us to introduce "yes-or-no" decisions into our linear programs and to introduce special logical conditions.

Binary variables can help us formulate a yes-or-no decision problem

In the written formulation of a linear program, binary variables are usually defined using the following form:

$$Y = \begin{cases} 1 \; if \; some \; condition \; holds \\ 0 \; otherwise \end{cases}$$

Sometimes we designate decision variables as binary if we are making a yes-or-no decision; for example, "Should we undertake this particular project?" "Should we buy that machine?" or "Should we locate a facility in Arkansas?" Other times, we create binary variables to introduce additional logic into our programs.

Techniques to Build Constrains with Binary Variables

One company have $b to invest for n projects. The investment of ith project needs \$a$_j$, and the profit is \$c$_j$, constraints are:

1) If project 1 is chosen, then project 2 must be chosen
2) at least choose two from Project 3,4,5
3) Must exactly choose one from either 6 or 7.

How to maximize the profits

$$x_i = \begin{cases} 0 \; invest\ on\ project\ i \\ 1 \; not\ invest\ on\ project\ i \end{cases}$$

max $\text{Z} = \sum_{i=1}^{n} c_j x_i$

s.t.

$\sum_{i=1}^{n} a_i x_i \leq b$

$x_1 \geq x_2$

$x_3 + x_4 + x_5 \geq 2$

$x_6 + x_7 = 1$

$x_i = 0,1 \; \forall i = 1, \dots, n$

1. Limiting the Number of Alternatives Selected

One common use of 0-1 variables involves limiting the number of projects or items that are selected from a group. Suppose a firm is required to select no more than two of three potential projects. This could be modeled with the following constraint:

$$\text{Y1} + \text{Y2} + \text{Y3} \leq 2$$

If we wished to force the selection of *exactly* two of the three projects for funding, the following constraint should be used:

$$\text{Y1} + \text{Y2} + \text{Y3} = 2$$

This forces exactly two of the variables to have values of 1, whereas the other variable must have a value of 0.

2. Dependent Selections

At times the selection of one project depends in some way on the selection of another project. This situation can be modeled with the use of 0-1 variables. Suppose G.E.'s new catalytic converter could be purchased ($Y_1 = 1$) only if the software was also purchased ($Y_2 = 1$). The following constraint would force this to occur:

$$\text{Y1} \leq \text{Y2}$$

or, equivalently,

$$\text{Y1} - \text{Y2} \leq 0$$

Thus, if the software is not purchased, the value of Y_2 is 0, and the value of Y_1 must also be 0 because of this constraint. However, if the software is purchased ($Y_2 = 1$), then it is possible that the catalytic converter could also be purchased ($Y_1 = 1$), although this is not required.
If we wished for the catalytic converter and the software projects to either both be selected or both not be selected, we should use the following constraint:

$$\mathrm{Y1} = \mathrm{Y2}$$

or, equivalently,

$$\mathrm{Y1} - \mathrm{Y2} = 0$$

Thus, if either of these variables is equal to 0, the other must also be 0. If either of these is equal to 1, the other must also be 1.

Assignment Problem

The coach of a swim team needs to assign swimmers to a 200-yard medley relay team (four swimmers, each swims 50 yards of one of the four strokes). Since most of the best swimmers are very fast in more than one stroke, it is not clear which swimmer should be assigned to each of the four strokes. The five fastest swimmers and their best times (in seconds) they have achieved in each of the strokes (for 50 yards) are

	Backstroke	Breaststroke	Butterfly	Freestyle
Carl	37.7	43.4	33.3	29.2
Chris	32.9	33.1	28.5	26.4
David	33.8	42.2	38.9	29.6
Tony	37.0	34.7	30.4	28.5
Ken	34.4	41.8	32.8	31.1

How should the swimmers be assigned to make the fastest relay team?

Formulation:

Person: i (from 1 to 5)

Style: j (from 1 to 4)

Decision variable:

$$x_{ij} = \begin{cases} 1 & \text{If person i takes style j} \\ 0 & Otherwise \end{cases}$$

i and j are just notations, not decision variables

Constraints:

$\sum_{j=1}^{4} x_{ij} \leq 1$ ∀i, which means each person can at most take one style

$\sum_{i=1}^{5} x_{ij} = 1$ ∀j, which means each style is exactly arranged one person

$x_{ij} = 0,1$

Objective Function:

Min Z $= \sum_i \sum_j c_{ij} x_{ij}$, where c_{ij} means time